The local integro cubic splines and its approximation properties

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Abstract

It is shown that using B-representation of splines, the construction of the integro cubic splines proposed by H.Behforooz [1] leads to the local integro cubic splines. The approximation properties of the local splines are also considered.

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Key words: Integro cubic splines, B-representation, approximation properties

Introduction

In [1], Behforooz.H first introduced a motivation of integro cubic splines to approximate the function and discussed the problem with a variety of the end conditions. He showed that when the first derivative representation and first derivative end conditions are used the construction of splines leads to the solution of tridiagonal system of linear equations. But additional so called third end condition that seems to be unnatural is needed. He pointed out that to construct the integro cubic splines in terms of the second derivative with any end conditions, one had to solve a system of linear equations with a full matrix of higher order. In [2] we showed that using B-representation of splines one can overcome the above mentioned difficulties arising in Behforooz's approach.

In this paper we will show another advantage of using B-representation of integro cubic splines.

For the integro cubic splines construction we do not have not only the function values, but also the derivatives values at knots. Therefore we can not use the end conditions that contains that information. In this paper we will show that it is possible to construct the local integro cubic splines without any end conditions. The coefficients of B- representation of these splines are determined explicitly by the means of given I_i.

1 The statement of problem

Suppose that the interval $[a, b]$ is partitioned by the following $k+1$ equally spaced points:

$$a = x_0 < x_1 < \cdots < x_k = b, \tag{1}$$

such that $x_i = a + ih$ for $i = 0, 1, \ldots, k$ with $h = (b-a)/k$.

Assume that the function values $y_i = y(x_i)$ are not given but are known integrals of $y = y(x)$ on k intervals $[x_{i-1}, x_i]$ and they are equal to

$$\int\limits_{x_{i-1}}^{x_i} y(x)dx = I_i, \quad i = 1(1)k. \tag{2}$$

The cubic splines $S(x) \in C^2[a, b]$ are called integro cubic ones [1], if

$$\int\limits_{x_{i-1}}^{x_i} S(x)dx = \int\limits_{x_{i-1}}^{x_i} y(x)dx = I_i, \quad i = 1(1)k. \tag{3}$$

We will use B-representation of cubic splines $S(x)$ of class $C^2[a,b]$. To do this the partition of $[a,b]$ is extended to the left and right sides by equally spaced knots

$$x_{-3} < x_{-2} < x_{-1} < x_0, \qquad x_k < x_{k+1} < x_{k+2} < x_{k+3}.$$

Then we have [3]

$$S(x) = \sum_{j=-1}^{k+1} \alpha_j B_j(x), \tag{4}$$

where $B_j(x)$ is a normalized cubic B-splines with compact support $[x_{j-2}, x_{j+2}]$. The coefficients in expansion (4) are determined as [4]

$$\alpha_{-1} = S_0 - h_0 m_0 + \frac{h_0^2}{3} M_0,$$
$$\alpha_j = S_j + \frac{h_j - h_{j-1}}{3} m_j - \frac{h_j h_{j-1}}{6} M_j, \qquad j = 0, 1, \ldots, k,$$
$$\alpha_{k+1} = S_k + h_{k-1} m_k + \frac{h_{k-1}^2}{3} M_k,$$

where $S_j = S(x_j)$, $\;m_j = S'(x_j)\; M_j = S''(x_j)$. In case of the uniform partition the last formula becomes

$$\alpha_j = S_j - \frac{h^2}{6} M_j, \qquad j = 0, 1, \ldots, k \tag{5}$$

and the values of splines $S(x)$ and its derivatives are determined by

$$S_i = \frac{\alpha_{i+1} + 4\alpha_i + \alpha_{i-1}}{6}, \tag{6a}$$

$$m_i = \frac{\alpha_{i+1} - \alpha_{i-1}}{2h}, \tag{6b}$$

$$M_i = \frac{\alpha_{i+1} - 2\alpha_i + \alpha_{i-1}}{h^2}, \tag{6c}$$

where $i = 0, 1, \ldots, k$.

2 Construction of local integro-cubic splines

In order to construct the integro cubic splines we first use the relations

$$
\begin{aligned}
&-\frac{h^3}{24}(M_{i-1}+M_i)+\frac{h}{2}(S_{i-1}+S_i)=I_i, \quad i=1(1)k, \\
&-\frac{h^3}{24}(M_i+M_{i+1})+\frac{h}{2}(S_i+S_{i+1})=I_{i+1}, \quad i=0(1)k-1,
\end{aligned}
\tag{7}
$$

which follows from (3). Substituting (6a) into (7) we obtain

$$\alpha_{i-1}+\alpha_i+\frac{h^2}{12}(M_{i-1}+M_i)=\frac{2}{h}I_i, \quad i=1(1)k, \tag{8}$$

$$\alpha_i+\alpha_{i+1}+\frac{h^2}{12}(M_i+M_{i+1})=\frac{2}{h}I_{i+1}, \quad i=0(1)k-1. \tag{9}$$

By adding (8) and (9) we have

$$\alpha_{i-1}+2\alpha_i+\alpha_{i+1}+\frac{h^2}{12}(M_{i-1}+2M_i+M_{i+1})=\frac{2}{h}(I_i+I_{i+1}), \quad i=1(1)k-1. \tag{10}$$

Substituting $M_{i+1}+M_{i-1}=2M_i+h(S'''_{i+0}-S'''_{i-0})$ and (6c) into (10) we obtain

$$\alpha_{i-1}+\alpha_i+\alpha_{i+1}=\frac{3}{2h}(I_i+I_{i+1})-\frac{h^3}{16}(S'''_{i+0}-S'''_{i-0}), \quad i=1(1)k-1. \tag{11}$$

The last term in the right hand side of (11) is small, because of $S'''_{i+0}-S'''_{i-0}=O(h)$. Ignoring this small quantity we have an approximate formulae

$$\alpha_{i-1}+\alpha_i+\alpha_{i+1}=\frac{3}{2h}(I_i+I_{i+1}), \quad i=1(1)k-1. \tag{12}$$

By (6c), the expression (10) can be rewritten in term of coefficients α_i as:

$$\alpha_{i-2}+12\alpha_{i-1}+22\alpha_i+12\alpha_{i+1}+\alpha_{i+2}=\frac{24}{h}(I_i+I_{i+1}), \tag{13}$$

where $i=1(1)k-1$. For convenience, we rewrite (13) in the following form

$$\alpha_{i-2}+\alpha_{i-1}+\alpha_i+11(\alpha_{i-1}+\alpha_i+\alpha_{i+1})+\alpha_i+\alpha_{i+1}+\alpha_{i+2}+9\alpha_i=\frac{24}{h}(I_i+I_{i+1}).$$

Using (12), from the last expression we can find α_i and it determined by formulae

$$\alpha_i = \frac{1}{6h}\{-I_{i-1} + 4I_i + 4I_{i+1} - I_{i+2}\}, \quad i = 2(1)k-2. \tag{14}$$

Thus we have explicit formula (14) for the coefficients of integro cubic splines. The remainder coefficients α_j of splines (4) can be found from the following relations

$$\alpha_{i-1} + 11\alpha_i + 11\alpha_{i+1} + \alpha_{i+2} = \frac{24}{h}I_{i+1}, \quad i = 0(1)k-1, \tag{15}$$

$$\alpha_{i-2} + 11\alpha_{i-1} + 11\alpha_i + \alpha_{i+1} = \frac{24}{h}I_i, \quad i = 1(1)k, \tag{16}$$

obtained after substituting (6c) into (8) and (9) respectively. In fact, from (15) for $i = 2$ one can find α_1 as

$$\alpha_1 = \frac{24}{h}I_3 - 11\alpha_2 - 11\alpha_3 - \alpha_4 \tag{17}$$

and from (15) for $i = 1$ and $i = 0$ one can find α_0 and α_{-1} as

$$\alpha_0 = \frac{24}{h}I_2 - 11\alpha_1 - 11\alpha_2 - \alpha_3 \,\text{and}\, \alpha_{-1} = \frac{24}{h}I_1 - 11\alpha_0 - 11\alpha_1 - \alpha_2. \tag{18}$$

Analogously one can find the coefficients α_{k-1}, α_k and α_{k+1} from (16) as

$$\begin{aligned} \alpha_{k-1} &= \frac{24}{h}I_{k-2} - 11\alpha_{k-2} - 11\alpha_{k-3} - \alpha_{k-4}, \\ \alpha_k &= \frac{24}{h}I_{k-1} - 11\alpha_{k-1} - 11\alpha_{k-2} - \alpha_{k-3} \text{ and} \\ \alpha_{k+1} &= \frac{24}{h}I_k - 11\alpha_k - 11\alpha_{k-1} - \alpha_{k-2}. \end{aligned} \tag{19}$$

Thus all the coefficients α_j of splines (4) are determined uniquely by explicit formulae (14) and (17)-(19). From (14) we see that the coefficients α_j of splines (4) are founded locally. That is α_i is determined only by $I_{i-1}, \;\; I_i, \;\; I_{i+1}$ and I_{i+2}. As a result, we have local cubic splines (4) and we call it local integro cubic splines. For our local integro-cubic splines one can prove the following approximation theorem.

Theorem 1. *Let y be a function of class $C^4[a,b]$ and $S(x)$ is a local integro cubic splines* (4)*, the coefficients of which are given by* (14) *and* (17)-(19)*. Then holds estimation*

$$||S^{(r)}(x) - y^{(r)}(x)||_\infty = O(h^{4-r}), \quad r = 0, 1, 2, 3, \tag{20}$$

where $||\circ||_\infty$ *is a uniform norm, i.e.,*

$$||\circ||_\infty = \max_{x\in[a,b]} |\circ|.$$

Proof. Using Taylor expansions of smooth function $y(x)$ around the point x_i we get

$$I_{i-1} = hy_i - \frac{3}{2}h^2y_i' + \frac{7}{6}h^3y_i'' - \frac{15}{24}h^4y''' + O(h^5),$$

$$I_{i+2} = hy_i + \frac{3}{2}h^2y_i' + \frac{7}{6}h^3y_i'' + \frac{15}{24}h^4y_i''' + O(h^5)$$

and

$$I_i + I_{i+1} = 2hy_i + \frac{h^3}{3}y_i'' + O(h^5).$$

Substituting these expansions into (14), we obtain

$$\alpha_i = y_i - \frac{h^2}{6}y_i'' + O(h^4), \quad i = 2(1)k-2. \tag{21}$$

Using Taylor expansion technique and taking into account (21) in (15) we get

$$\alpha_0 = y_0 - \frac{h^2}{6}y_0'' + O(h^4), \quad \alpha_1 = y_1 - \frac{h^2}{6}y_1'' + O(h^4) \tag{22}$$

and

$$\alpha_{-1} = y_0 - hy_0' + \frac{h^3}{3}y_0'' + O(h^4). \tag{23}$$

Analogously one can find approximate expressions for the coefficients α_{k-1}, α_k and α_{k+1}

$$\alpha_i = y_i - \frac{h^2}{6}y_i'' + O(h^4), \quad i = k-1, k, \tag{24}$$

$$\alpha_{k+1} = y_k + hy_k' + \frac{h^2}{3}y_k'' + O(h^4). \tag{25}$$

It is known [4] that the estimation kind of (20) is valid for a quasiinterpolatory cubic splines $\tilde{S}$ given by

$$\tilde{S} = \sum_{j=-1}^{k+1} \tilde{\alpha_j} B_j(x), \tag{26}$$

where

$$\tilde{\alpha_j} = y_j - \frac{h^2}{6} y_j'', \quad j = 0(1)k \tag{27}$$

and

$$\tilde{\alpha}_{-1} = y_0 - hy_0' + \frac{h^2}{3} y_0'', \tag{28}$$

$$\tilde{\alpha}_{k+1} = y_k + hy_k' + \frac{h^2}{3} y_k''. \tag{29}$$

From (21)-(25) and (27)-(29) it clear that

$$\alpha_i - \tilde{\alpha}_i = O(h^4), \quad i = -1, 0, 1, \ldots, k, k+1. \tag{30}$$

Then from (4), (26) and (30) it follows

$$S - \tilde{S} = O(h^4).$$

Using the triangular inequality

$$||S - y|| \leq ||S - \tilde{S}|| + ||\tilde{S} - y||$$

gives estimation (20) for $r = 0$. The remainder estimate for $r = 1, 2$ and 3 obviously follows from (30). This completes the proof of theorem. □

The direct consequence of formulae (21)-(25) is the next equalities

$$S_i - y_i = \frac{h^2}{6}(M_i - y_i'') + O(h^4), \quad i = 0(1)k, \tag{31}$$

$$m_i - y_i' = O(h^3), \qquad i = 0(1)k. \tag{32}$$

The Taylor expansion technique also gives us

$$\alpha_i = \hat{\alpha}_i + O(h^4), \quad i = -1, 0, \ldots, k, k+1,$$

where $\hat{\alpha_i}$ are the coefficients of locally interpolatory cubic splines [4] i.e.,

$$\hat{\alpha}_i = \frac{8y_i - y_{i-1} - y_{i+1}}{6}, \quad i = 1(1)k - 1.$$

Thus the local integro and locally interpolatory cubic splines S and $\hat{S}$ has the same approximation properties.

3 Numerical examples

In this section, we present results of the numerical experiment to illustrate the approximation properties of the local integro cubic splines. Suppose that $y(x) \in C^4[0,1]$, we consider the following test functions

$$y_1(x) = \exp(x), \quad y_2(x) = \cos(\pi x).$$

The results are shown in the table

	$\lvert S_j - y_{1,j}\rvert$			$\lvert m_j - y'_{1,j}\rvert$			$\lvert M_j - y''_{1,j}\rvert$		
x_j	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$
0	2.85E-03	1.56E-04	9.15E-06	1.39E-01	1.53E-02	1.79E-03	3.40E+0	7.46E-01	1.75E-01
0.1	2.93E-04	1.94E-06	3.84E-08	1.41E-02	1.54E-04	4.08E-08	3.38E-01	5.91E-03	5.76E-05
0.2	3.56E-05	6.79E-07	4.24E-08	1.41E-03	7.21E-07	4.51E-08	2.69E-02	2.55E-04	6.37E-05
0.3	1.20E-05	7.50E-07	4.69E-08	1.28E-05	7.97E-07	4.98E-08	1.14E-03	2.82E-04	7.04E-05
0.4	1.33E-05	8.29E-07	5.18E-08	1.41E-05	8.81E-07	5.50E-08	1.26E-03	3.12E-04	7.77E-05
0.5	1.47E-05	9.17E-07	5.73E-08	1.56E-05	9.74E-07	6.08E-08	1.39E-03	3.44E-04	8.59E-05
0.6	1.62E-05	1.01E-06	6.33E-08	1.72E-05	1.08E-06	6.72E-08	1.53E-03	3.81E-04	9.50E-05
0.7	1.79E-05	1.12E-06	6.99E-08	1.91E-05	1.19E-06	7.43E-08	1.70E-03	4.21E-04	1.05E-04
0.8	5.68E-05	1.24E-06	7.73E-08	2.28E-03	1.31E-06	8.21E-08	4.41E-02	4.65E-04	1.16E-04
0.9	4.74E-04	4.05E-06	8.54E-08	2.28E-02	3.24E-04	9.08E-08	5.49E-01	1.25E-02	1.28E-04
1	4.61E-03	3.28E-04	2.18E-05	2.26E-01	3.20E-02	4.27E-03	5.51E+0	1.57E+0	4.17E-01

	$\lvert S_j - y_{2,j}\rvert$			$\lvert m_j - y'_{2,j}\rvert$			$\lvert M_j - y''_{2,j}\rvert$		
x_j	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$	$k=10$	$k=20$	$k=40$
0	6.21E-04	4.05E-05	2.55E-06	3.11E-02	4.05E-03	5.11E-04	8.44E-01	2.19E-01	5.51E-02
0.1	7.70E-05	3.97E-07	5.02E-08	3.10E-03	4.38E-05	2.07E-07	2.31E-14	2.14E-02	4.83E-03
0.2	5.33E-06	6.94E-07	4.30E-08	3.93E-04	5.11E-06	2.93E-07	7.44E-02	1.65E-02	4.11E-03
0.3	8.41E-06	5.09E-07	3.13E-08	7.15E-05	6.46E-06	4.03E-07	4.66E-02	1.20E-02	2.99E-03
0.4	4.74E-06	2.71E-07	1.65E-08	1.25E-04	7.59E-06	4.74E-07	2.59E-02	6.32E-03	1.57E-03
0.5	4.80E-07	7.74E-09	1.22E-10	1.27E-04	7.98E-06	4.98E-07	5.76E-04	3.72E-05	2.34E-06
0.6	3.78E-06	2.56E-07	1.62E-08	1.25E-04	7.59E-06	4.74E-07	2.47E-02	6.24E-03	1.57E-03
0.7	9.37E-06	4.94E-07	3.10E-08	7.13E-05	6.46E-06	4.03E-07	4.77E-02	1.19E-02	2.98E-03
0.8	4.34E-06	6.78E-07	4.27E-08	3.94E-04	5.11E-06	2.93E-07	7.32E-02	1.64E-02	4.10E-03
0.9	7.83E-05	3.81E-07	5.00E-08	3.12E-03	4.38E-05	2.07E-07	7.68E-04	2.13E-02	4.83E-03
1	6.25E-04	4.05E-05	2.56E-06	3.13E-02	4.05E-03	5.11E-04	8.47E-01	2.19E-01	5.51E-02

As shown in the table, the approximation properties of local integro cubic splines were confirmed by numerical experiments.

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References

[1] H.Behforooz, *Approximation by integro cubic splines,* Appl. Math. Comput. 175(2006), 8-15

[2] T.Zhanlav, R.Mijiddorj, *Integro cubic splines and their approximation properties.* Applied Mathemetics Series, Tver State University, Russia. 26(86)(2008),65-77

[3] Yu.S.Zav'yalov., B.I.Kvasov and V.L.Miroschnichenko, *Spline function methods*, M, 1980 (in Russian)

[4] T.Zhanlav, *B-representation of cubic interfolating splines, Vychislitel'nye Systems*, N87, Novisibirsk, 1981, 3-10 (in Russian)